Phase Models and Phase Computations for Oscillators

Alper Demir and Önder Şuvak

Koç University, Istanbul, Turkey aldemir@ku.edu.tr, osuvak@ku.edu.tr

Summary. Oscillators have been a research focus in many disciplines such as electronics and biology. The time keeping capability of oscillators is best described by the scalar quantity called *phase*. Phase computations and equations describing phase dynamics have been useful in understanding oscillator behavior and designing oscillators least affected by disturbances such as noise. In this talk, we present a unified review of phase models for oscillators assimilating the work that has been done in electronics and biology for the last seven decades.

Oscillatory behavior is seen in physical and man-made systems, where their time keeping ability is important. Oscillators are particularly encountered in or introduced into biological and electronic systems where the adverse effects of disturbances such as noise degrade their time keeping and synchronization capability.

The dynamical behavior of oscillators is best described and analyzed in terms of the scalar quantity, *phase*. Of the pertaining notions in the literature, the most straightforward phase definition is obtained when a planar oscillator is expressed in polar coordinates, with amplitude and polar angle as the state variables. The usefulness of the polar angle as phase does not generalize to higher dimensional oscillators. In the general case, it is our conviction that the most rigorous and precise definition of phase is the one based on the so-called *isochrons* of an oscillator [1–4]. The notion of isochrons was first proposed by Arthur Winfree [1, 3] in 1974, who has also coined the term. It was later revealed that isochrons are intimately related to the notion of asymptotic phase in the theory of differential equations [5, 6]. The isochron theoretic phase of a free-running oscillator is simply time itself. Such an unperturbed oscillator serves as a perfect time keeper if it is in the process of converging to a limit cycle, even when it has not yet settled to a periodic steady-state solution. Perturbations make the actual phase deviate from time, due to the degrading impact of disturbances on the time keeping ability.

Phase is a quantity that compactly describes the dynamical behavior of an oscillator. One is then interested in computing the phase of a perturbed oscillator. If this can be done in a semi or fully analytical manner for a practical oscillator, one can draw conclusions and obtain useful characterizations in assessing the time keeping performance. Indeed, we observe in the literature that, in various disciplines, researchers have derived *phase equations* that compactly describe the dynamics of weakly perturbed oscillators [2, 7]. It appears that a phase equation for oscillators has first been derived by Malkin [8] in his work on the reduction of weakly perturbed oscillators to their phase models [2], and the same equation has been subsequently reinvented by various other researchers in several disciplines [1, 7, 9]. This phase equation has been used in mathematical biology to study circadian rhythms and coupled oscillators in models of neurological systems [1, 2, 10], and in electronics for the analysis of phase noise and timing jitter in oscillators [7, 11, 12]. The acclaimed phase equation is a nonlinear but *scalar* differential equation. As such, it is the ultimate *reduced-order* model for a complex nonlinear dynamical system. Its scalar nature and the specific form of the nonlinearity in this equation makes it possible in some cases to solve, or characterize the solutions of, this equation in (semi) analytical form, e.g., in the investigation of synchronization of coupled oscillators [2, 9] and in characterizing phase noise in electronic oscillators with stochastic perturbations as models of electronic noise sources [7, 13].

In this talk, we present a unified review of phase models for autonomous oscillators assimilating the work that has been done on oscillator analysis in both electronics and mathematical biology during the past seventy years. We first review the notion of isochrons, which forms the basis for the generalized phase notion for an oscillator. We then present an overview of techniques for computing local approximations for the isochrons of an oscillator [4, 14]. Next, we describe phase models and phase computation schemes based on local approximations of isochrons, for continuous periodic (single-frequency) oscillators [15], continuous quasi-periodic (multi-frequency) oscillators [16], as well as for discrete molecular oscillators [17, 18].

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