Transmission line parameters computed by NURBS-based impedance boundary conditions: the case of different conductivities

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Summary. High order surface impedance boundary conditions (SIBCs) have been coupled with the Boundary Element Method (BEM) to produce an integral formulation for the computation of the impedance matrix of multiconductor transmission lines of arbitrary cross-section [1, 2]. The method extends the use of SIBCs into lower frequencies and does so efficiently in that the solution of the integral equations need only be computed once whereas the solution may be obtained over the whole applicable frequency domain. In the case of different conductivities of the parallel conductors, care must be taken in the perturbation expansion when the formulation is derived. The use of NURBS gives a better representation of complex geometries and helps in the computation of the radius of curvature and of the tangential derivatives of the unknowns. As a realistic application, the per-unit-length parameters of sector shaped cables are computed, showing the accuracy of the method.

1 Non-uniform rational B-splines

Recently, the so-called Isogeometric Analysis method was introduced in the context of mechanical engineering [3], with the aim of improving the communication between Computer Aided Design (CAD) software and numerical solvers. The method can be understood as a generalization of finite elements, where the standard polynomial shape functions are replaced by the functions used by CAD to describe the geometry.

The most widespread functions in CAD are probably non-uniform rational B-splines (NURBS), due to their flexibility and their capability to design smooth geometries. To define a NURBS curve first it is necessary to introduce a partition of a reference interval. NURBS basis functions are defined on this partition as a set of piecewise rational polynomials. The curve is then created as a linear combination of these basis functions, by associating a control point to each one of them [4].

The method we propose is based on NURBS to represent the contour of the cross section of the conductors, whereas the discrete solution is sought as a non-rational spline. The use of NURBS not only gives a good representation of complex geometries, but it also allows an exact computation of the radius of curvature, as required by high order SIBCs. Moreover, a discretization based on high order B-splines is necessary to compute the tangential derivatives appearing in high order SIBCs, which can not be accurately computed with low order BEM.

2 Integral formulation of the problem

We work on a two-dimensional geometry. Assume that we have N different conductors, where electric currents of intensity I_j, $j = 1,\dots,N$ flow. We denote by Γ_j the boundary of their cross sections. We choose an eddy-current model written in terms of the magnetic vector potential $\mathbf{A}$. In the 2D case this vector is parallel to the conductors axis, for which $\mathbf{A} = A\mathbf{e_z}$.

Splitting the potential into "source" and "eddy" components, $A = A^s + A^e$, our continuous problem becomes

$$\Delta A^e = \mathrm{i}\omega\mu\sigma A^e,\ \int_{\Gamma_j} \frac{1}{\mu}\frac{\partial A^e}{\partial n} = I_j, \tag{1}$$
$$[A^e]_{\Gamma_j} = -A^s,\ \left[\frac{\partial A^e}{\partial n}\right]_{\Gamma_j} = 0,\ A^e = O\left(\frac{1}{|\mathbf{r}|}\right), |\mathbf{r}| \to +\infty,$$

where A^s is an unknown, and is constant for each conductor.

Denoting by $G(\mathbf{r},\mathbf{r}')$ the fundamental solution of the 2D Laplace equation, we define the integral operators associated to the single and double layer potentials

$$S_j u(\mathbf{r}) = \oint_{\Gamma_j} G(\mathbf{r},\mathbf{r}')u(\mathbf{r}')\,d\gamma(\mathbf{r}'), \tag{2}$$
$$D_j u(\mathbf{r}) = \oint_{\Gamma_j} \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n_{\mathbf{r}'}} u(\mathbf{r}')\,d\gamma(\mathbf{r}'), \tag{3}$$

and denoting by $K = \dfrac{\partial A^e_{\mathrm{ext}}}{\partial n}$, the solution of our problem satisfies the integral equation

$$A^s(\mathbf{r}) + \sum_{j=1}^{N} S_j K(\mathbf{r}) = \left(\frac{-I}{2} + \sum_{j=1}^{N} D_j\right) A^e_{\mathrm{int}}(\mathbf{r}). \tag{4}$$

3 Approximation by SIBCs

SIBCs can be applied whenever the skin depth

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}, \tag{5}$$

is "small enough". Following [1,5], the fields are written as asymptotic expansions in terms of δ, in the form:

$$A^e_{\mathrm{int}}(\mathbf{r},\delta) \simeq \sum_{i=0}^{3} A^{e,i}_{\mathrm{int}}(\mathbf{r})\delta^i, \tag{6}$$

$$A^s(\mathbf{r},\delta) \simeq \sum_{i=0}^{3} A^{s,i}(\mathbf{r})\delta^i,\ K(\mathbf{r},\delta) \simeq \sum_{i=0}^{3} K^i(\mathbf{r})\delta^i, \tag{7}$$

and denoting the curvature of Γ by $\mathscr{C}$, and by $\frac{\partial^2 u}{\partial \tau^2}$ the second tangential partial derivative, it holds:

$$A^{e,i}_{\mathrm{int}} = \sum_{l=1}^{i} \psi_l(K^{i-l}), \quad \text{with} \quad \psi_1[u] = u,$$
$$\psi_2[u] = \frac{\mathscr{C}}{2}u, \quad \psi_3[u] = \frac{3\mathscr{C}^2}{8}u + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}. \tag{8}$$

We then solve sequentially the problem, for $i = 0,\dots,3$

$$A^{s,i}(\mathbf{r}) + \sum_{j=1}^{N} S_j K^i(\mathbf{r}) =$$
$$\left(\frac{-I}{2} + \sum_{j=1}^{N} D_j\right)\left(\sum_{l=1}^{i} \psi_l(K^{i-l}(\mathbf{r}))\right), \tag{9}$$

together with the intensity conditions: $\int_{\Gamma_j} \frac{1}{\mu} K^0 = I_j$, and $\int_{\Gamma_j} \frac{1}{\mu} K^l = 0$, for $l = 1,2,3$.

4 The case of different conductivities

Let us assume that the electrical conductivity of each conductor is σ_j, $j = 1,\dots,N$, and define the small parameter for each conductor, δ_j, as in (5). It is also necessary to define a small parameter for the exterior domain, that we can take, for instance, $\delta_0 = \delta_N$. With this choice of small parameters, we rewrite the asymptotic expansions (7) based on δ_0, whereas the expansion (6) is written with a different δ_j in each conductor. Since the small parameters are different for the conductors and the insulator, when considering the continuity conditions on the interface, it is not possible just to equate the terms with the same coefficients, but we must adjust the equations multiplying and dividing some terms by powers of δ_0.

5 Three sector-shaped cable

We have applied the method to the simulation of a three sector-shaped cable with a shield, as the one shown in Fig. 1. Each sector is made of copper, with $\sigma = 5.8 \times 10^7$ S/m, and for the shield the electrical conductivity is $\sigma = 1.1 \times 10^6$ S/m. We notice that the corners of each sector have been rounded, because the SIBCs can only be applied in smooth geometries.

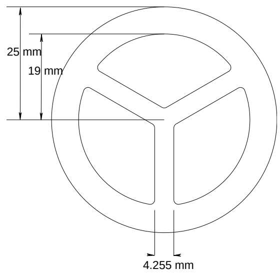

Fig. 1. Geometry of the three sector-shaped cable

The contour of each sector is parametrized with a quadratic NURBS with 9 elements, and the shield with a quadratic NURBS formed by 4 elements. Then, the problem is solved in a refined mesh formed by 45 elements on each conductor, and 20 elements on the shield. Our results are compared in Fig. 2 with the ones given by a commercial FEM software, in a mesh formed by 187607 elements.

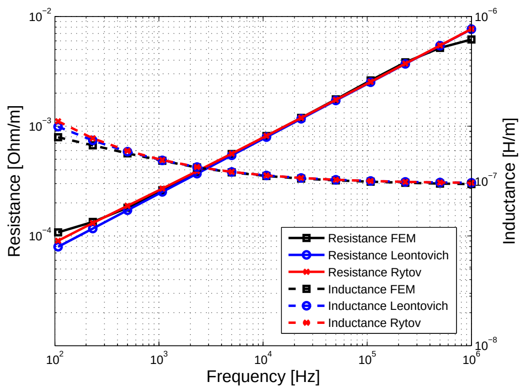

Fig. 2. P.u.l. self-resistance and self-inductance for one sector of the three sector cable

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