On an integral equation method for the electromagnetic scattering of biperiodic structures

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Summary. In this note we study an integral formulation for electromagnetic scattering by a biperiodic structure. It is derived from the time-harmonic Maxwell equations via potential methods by the combined use of a Stratton-Chu integral representation and an electric potential ansatz. We obtain results on existence and uniqueness for the solutions of this singular integral equation and give an outlook on the equation's numerical treatment via the fast multipole Boundary Element Method.

1 Introduction

Studying an integral formulation for electromagnetic scattering by a biperiodic structure generalizes the results from [5] where the equivalent problem for one-periodic structures was treated. Up to now, both in the one- and the biperiodic case several integral formulations have been proposed and implemented (e.g. [4]). We derive a new formulation by adapting the approach of [2], in which instead of a periodic structure a bounded obstacle was focussed on.

2 The electromagnetic scattering problem

Let Σ be a smooth non-selfintersecting surface which is 2π-periodic in both x_1- and in x_2-direction and separates two regions $G_\pm \subset \mathbb{R}^3$ filled with materials of constant electric permittivity $\varepsilon_\pm$ and magnetic permeability $\mu_\pm$. The surface is illuminated from G_+ by an electromagnetic plane wave at oblique incidence

$$(\mathbf{E}^{\mathrm{i}}, \mathbf{H}^{\mathrm{i}}) = (\mathbf{p}, \mathbf{s}) e^{\mathrm{i}(\alpha_1 x_1 + \alpha_2 x_2 - \alpha_3 x_3)} e^{-\mathrm{i}\omega t}, \qquad (1)$$

which is $\tilde{\alpha}$-quasiperiodic[1] in x_1 and in x_2 of period 2π, i.e. satisfies the relation

$$\mathbf{u}(\tilde{x} + 2\pi, x_3) = e^{\mathrm{i}2\pi(\alpha_1+\alpha_2)} \mathbf{u}(x). \qquad (2)$$

The total fields are given by

$$\mathbf{E}_+ = \mathbf{E}^{\mathrm{i}} + \mathbf{E}^{\mathrm{refl}}, \quad \mathbf{H}_+ = \mathbf{H}^{\mathrm{i}} + \mathbf{H}^{\mathrm{refl}}, \qquad (3)$$

$$\mathbf{E}_- = \mathbf{E}^{\mathrm{tran}}, \quad \mathbf{H}_- = \mathbf{H}^{\mathrm{tran}} \qquad (4)$$

[1] In the following the tilde indicates the orthogonal projection of a three-dimensional vector on the (x_1, x_2)-plane.

and - after dropping the factor $e^{-i\omega t}$ - satisfy the time-harmonic Maxwell equations

$$\operatorname{curl} \mathbf{E} = \mathrm{i}\omega\mu\mathbf{H} \quad \text{and} \quad \operatorname{curl} \mathbf{H} = -\mathrm{i}\omega\varepsilon\mathbf{E}, \qquad (5)$$

just like the incident and the scattered fields. When crossing the surface the tangential components of the total fields are continuous

$$\mathbf{n} \times (\mathbf{E}_+ - \mathbf{E}_-) = 0, \quad \text{on } \Sigma, \qquad (6)$$

$$\mathbf{n} \times (\mathbf{H}_+ - \mathbf{H}_-) = 0, \quad \text{on } \Sigma, \qquad (7)$$

where $\mathbf{n}$ is the unit normal to the interface Σ. As the domain is unbounded, we must additionally impose the so called outgoing wave condition at infinity

$$(\mathbf{E}^{\mathrm{refl}}, \mathbf{H}^{\mathrm{refl}}) = \sum_{n \in \mathbb{Z}^2} (\mathbf{E}_n^+, \mathbf{H}_n^+) e^{\mathrm{i}(\alpha_n \cdot \tilde{x} + \beta_n^+ x_3)}, \qquad (8)$$

$$(\mathbf{E}^{\mathrm{tran}}, \mathbf{H}^{\mathrm{tran}}) = \sum_{n \in \mathbb{Z}^2} (\mathbf{E}_n^-, \mathbf{H}_n^-) e^{\mathrm{i}(\alpha_n \cdot \tilde{x} - \beta_n^- x_3)}, \qquad (9)$$

where $n = (n_1, n_2)^{\mathrm{T}}$, $\tilde{x} = (x_1, x_2)^{\mathrm{T}}$, $\alpha_n = (\alpha_1 + n_1, \alpha_2 + n_2)$ and $\beta_n^\pm = \sqrt{\kappa_\pm^2 - |\alpha_n|^2}$ with $\kappa_\pm^2 = \omega^2 \varepsilon_\pm \mu_\pm$. We shall assume $\kappa_+ > 0$, $\operatorname{Re}\kappa_- > 0$, $\operatorname{Im}\kappa_- \geq 0$. As we can easily derive the magnetic field in dependence of the electric field $\mathbf{E}$ as $\mathbf{H} = -\frac{\mathrm{i}}{\omega\mu} \operatorname{curl} \mathbf{E}$, we are now interested in finding vector fields $\mathbf{E}$ satisfying (5)-(8) such that

$$\mathbf{E},\ \operatorname{curl} \mathbf{E} \in (L^2_{loc}(\mathbb{R}^3))^3. \qquad (10)$$

The $\tilde{\alpha}$-quasiperiodicity of the incident waves motivates these two fields to be $\tilde{\alpha}$-quasiperiodic themselves.

3 Boundary integral formulation

In order to solve the electromagnetic scattering problem introduced in section 2, we derive an equivalent integral equation via potential methods. For this, we combine a direct with an indirect method: in the domain G_+ above the grating surface Σ, we work with the quasiperiodic version of the Stratton-Chu integral representation and in the domain G_- below the grating surface, we make use of an electric potential ansatz. As it is common when working with periodic structures, we restrict our calculations to one period $\Gamma = \{\tilde{x} \mid 0 < x_1, x_2 < 2\pi\}$ of the surface. Its one-sided limit from $G_\pm$ will be denoted by $\Gamma_\pm$.

3.1 Derivation of the boundary integral equation

The potentials which provide $\tilde{\alpha}$-quasiperiodic solutions of the time-harmonic Maxwell equations are based on the $\tilde{\alpha}$-quasiperiodic fundamental solution

$$G_{\kappa,\tilde{\alpha}}(x) = \frac{i}{8\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{e^{i\alpha_n\cdot\tilde{x}+i\beta_n|x|}}{\beta_n}. \qquad (11)$$

The single layer potential $\mathbf{S}_{\kappa,\tilde{\alpha}}$ is then given by

$$(\mathbf{S}_{\kappa,\tilde{\alpha}}\mathbf{u})(x) = \int_\Gamma G_{\kappa,\tilde{\alpha}}(x-y)\mathbf{u}(y)\, d\sigma(y), \qquad (12)$$

for $x \in \mathbb{R}^3 \setminus \Gamma$. We define the electric potential $\Psi_{E_\kappa}^{\tilde{\alpha}}$ generated by $\mathbf{j} \in \mathbf{H}_{\times,\tilde{\alpha}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ as

$$\Psi_{E_\kappa}^{\tilde{\alpha}}\mathbf{j} = \kappa^{-1}\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{S}_{\kappa,\tilde{\alpha}}\mathbf{j} \qquad (13)$$

and the magnetic potential $\Psi_{M_\kappa}^{\tilde{\alpha}}$ generated by $\mathbf{m} \in \mathbf{H}_{\times,\tilde{\alpha}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ as

$$\Psi_{M_\kappa}^{\tilde{\alpha}}\mathbf{m} = \mathrm{curl}\,\mathbf{S}_{\kappa,\tilde{\alpha}}\mathbf{m}. \qquad (14)$$

Defining the Dirichlet traces γ_{D} and the Neumann traces $\gamma_{\mathrm{N}_\kappa}$

$$\gamma_{\mathrm{D}}^{\pm}\mathbf{u} = (\mathbf{n}\times\mathbf{u})\,|_{\Gamma_\pm}, \gamma_{\mathrm{N}_\kappa}^{\pm}\mathbf{u} = \kappa^{-1}\,(\mathbf{n}\times\mathrm{curl}\,\mathbf{u})\,|_{\Gamma_\pm} \quad (15)$$

as well as

$$[\gamma_{\mathrm{D}}] = \gamma_{\mathrm{D}}^{-} - \gamma_{\mathrm{D}}^{+}, \{\gamma_{\mathrm{D}}\} = -\frac{1}{2}\left(\gamma_{\mathrm{D}}^{-} + \gamma_{\mathrm{D}}^{+}\right), \qquad (16)$$

$$[\gamma_{\mathrm{N}_\kappa}] = \gamma_{\mathrm{N}_\kappa}^{-} - \gamma_{\mathrm{N}_\kappa}^{+}, \{\gamma_{\mathrm{N}_\kappa}\} = -\frac{1}{2}\left(\gamma_{\mathrm{N}_\kappa}^{-} + \gamma_{\mathrm{N}_\kappa}^{+}\right), \quad (17)$$

we have the following jump relations for the electric and magnetic potential:

$$[\gamma_{\mathrm{D}}]\Psi_{\mathrm{E}_\kappa}^{\tilde{\alpha}} = 0, [\gamma_{\mathrm{N}_\kappa}]\Psi_{\mathrm{E}_\kappa}^{\tilde{\alpha}} = -I, \qquad (18)$$

$$[\gamma_{\mathrm{N}_\kappa}]\Psi_{\mathrm{M}_\kappa}^{\tilde{\alpha}} = 0, [\gamma_{\mathrm{D}}]\Psi_{\mathrm{M}_\kappa}^{\tilde{\alpha}} = -I. \qquad (19)$$

With the Stratton-Chu ansatz

$$\mathbf{E}^{\mathrm{refl}} = \Psi_{\mathrm{E}_{\kappa_+}}^{\tilde{\alpha}}\,\gamma_{\mathrm{N}_{\kappa_+}}^{\tilde{\alpha}}\,\mathbf{E}^{\mathrm{refl}} + \Psi_{\mathrm{M}_{\kappa_+}}^{\tilde{\alpha}}\,\gamma_{\mathrm{D}}^{+}\mathbf{E}^{\mathrm{refl}} \qquad (20)$$

in G_+ and the ansatz

$$\mathbf{E}^{\mathrm{tran}} = \Psi_{\mathrm{E}_{\kappa_+}}^{\tilde{\alpha}}\mathbf{j} \qquad (21)$$

in G_-, the use of the transmission conditions (6),(7) as well as the use of the jump relations (18),(19) for the electric and magnetic potential lead to the singular integral equation

$$\mathbf{A}_{\tilde{\alpha}}\mathbf{j} = \left[\rho_1 C_{\tilde{\alpha}}^{+}\left(M_{\tilde{\alpha}}^{-} + \frac{1}{2}I\right) + \left(M_{\tilde{\alpha}}^{+} + \frac{1}{2}I\right)C_{\tilde{\alpha}}^{-}\right]\mathbf{j} = -\gamma_D^{-}\mathbf{E}^{\mathrm{i}}, \qquad (22)$$

where $\rho_1 = \frac{\mu_+\kappa_-}{\mu_-\kappa_+}$ and

$$C_{\tilde{\alpha}}^{\pm} = \{\gamma_{\mathrm{D}}\}\Psi_{\mathrm{E}_{\kappa_\pm}}^{\tilde{\alpha}} = \{\gamma_{\mathrm{N}_{\kappa_\pm}}\}\Psi_{\mathrm{M}_{\kappa_\pm}}^{\tilde{\alpha}}, \qquad (23)$$

$$M_{\tilde{\alpha}}^{\pm} = \{\gamma_{\mathrm{D}}\}\Psi_{\mathrm{M}_{\kappa_\pm}}^{\tilde{\alpha}} = \{\gamma_{\mathrm{N}_{\kappa_\pm}}\}\Psi_{\mathrm{E}_{\kappa_\pm}}^{\tilde{\alpha}}. \qquad (24)$$

3.2 Properties of the boundary integral equation

We can show that the singular integral operator $\mathbf{A}_{\tilde{\alpha}}$ is Fredholm with index 0 and that under certain conditions there exists a unique solution of the integral equation (22). The proofs are based on techniques used in [3], [2] and [5].

Theorem 1 (Fredholmness). *Assume that the electric permittivity $\varepsilon_\pm$ and the magnetic permeability $\mu_\pm$ satisfy $\left(1+\frac{\mu_-}{\mu_+}\right) \neq 0$ and $\left(1+\frac{\varepsilon_+}{\varepsilon_-}\right) \neq 0$. Then $\mathbf{A}_{\tilde{\alpha}}$ is a Fredholm operator of index zero on $\mathbf{H}_{\times,\tilde{\alpha}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$.*

Theorem 2 (uniqueness). *Assume $\mathrm{Im}\,\varepsilon_-, \mathrm{Im}\,\mu_- \geq 0$ with $\mathrm{Im}(\varepsilon_+ + \mu_+) \geq 0$. Then (22) has at most one solution if $\ker\{\Psi_{\mathrm{E}_{\kappa_+}}^{\tilde{\alpha}}\} = \{0\}$.*

Theorem 3 (existence). *Let $\varepsilon_-, \mu_- \in \mathbb{R}_+$ and suppose the conditions of Theorem 1 are satisfied. If the electric potential $\Psi_{\mathrm{E}_{\kappa_-}}^{\tilde{\alpha}}$ is invertible, then there exists a solution $\mathbf{j} \in \mathbf{H}_{\times,\tilde{\alpha}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ of (22).*

4 Numerical treatment and prospects

Considering the future implementation of the integral equation (22) we will use the Boundary Element Method which reduces the spatial dimensionality by one compared to the Finite Element Method. Furthermore, we want to accelerate occurring multiplications via a fast multipole method. A crucial issue is the evaluation of the $\tilde{\alpha}$-quasiperiodic Green's function (11). The use of Ewald's method seems to be promising in this context (cp. [1]).

So far we have only studied the electromagnetic scattering problem for smooth surfaces Σ, but want to extend our results to Lipschitz surfaces with edges and corners.

References

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