High-Order Local Time-Stepping with Explicit Runge-Kutta Methods

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Summary. We propose explicit local time-stepping (LTS) schemes of high accuracy based either on classical or low-storage Runge-Kutta schemes for time dependent Maxwell's equations. By using smaller time steps precisely where smaller elements in the mesh are located, these methods overcome the bottleneck caused by local mesh refinement in explicit time integrators.

1 FE Discretizations of Maxwell's Equations

The evolution of a time-dependent electromagnetic field $\mathbf{E}(\mathbf{x},t)$, $\mathbf{H}(\mathbf{x},t)$ propagating through a linear isotropic medium is governed by Maxwell's equations:

$$\varepsilon \mathbf{E}_t = \nabla \times \mathbf{H} - \sigma \mathbf{E} + \mathbf{j}, \tag{1}$$

$$\mu \mathbf{H}_t = \nabla \times \mathbf{E}. \tag{2}$$

Here the coefficients μ, ε and σ denote the relative magnetic permeability, the relative electric permittivity and the conductivity of the medium, respectively. The source term $\mathbf{j}$ corresponds to the applied current density.

We discretize (1)-(2) in space by using standard edge finite elements (FE) with mass lumping [6] or a discontinuous Galerkin (DG) FE discretization [4, 5], while leaving time continuous. Either discretization leads to a system of ordinary differential equations with an essentially diagonal mass matrix. Thus, when combined with explicit time integration, the resulting fully discrete scheme of (1)-(2) will be truly explicit.

2 Runge-Kutta based LTS

Locally refined meshes impose severe stability constraints on explicit time-stepping methods for the numerical solution of (1)-(2). Local time-stepping methods overcome that bottleneck by using smaller time-steps precisely where the smallest elements in the mesh are located. In [1, 2], explicit second-order LTS integrators for transient wave motion were developed, which are based on the standard leap-frog scheme. In the absence of damping, i.e. $\sigma = 0$, these time-stepping schemes, when combined with the modified equation approach, yield methods of arbitrarily high (even) order. By blending the leap-frog and the Crank-Nicolson methods, a second-order LTS scheme was also derived there for (damped) electromagnetic waves in conducting media, i.e. $\sigma > 0$, yet this approach cannot be readily extended beyond order two. To achieve arbitrarily high accuracy in the presence of damping, while remaining fully explicit, explicit LTS methods for the scalar damped wave equation based on Adams-Bashforth multi-step schemes were derived in [3].

Here we propose explicit LTS methods of high accuracy based either on explicit classical or low-storage Runge-Kutta (RK) schemes. In contrast to Adams-Bashforth methods, RK methods are one-step methods; hence, they do not require a starting procedure and easily accommodate adaptive time-step selection. Although, RK methods do require several further evaluations per time-step, that additional work is compensated by a less stringent CFL stability restriction.

Clearly, the idea of using different time-steps for different components in the context of ordinary differential equations is not new [7]. However, RK methods achieve higher accuracy not by extrapolating farther from the (known) past but instead by including further intermediate stages from the current time-step. Thus, for the numerical solution of partial differential equations, the derivation of high-order local time-stepping methods that are based on RK schemes, is generally more difficult.

3 Numerical Experiments

To illustrate the versatility of our approach, we consider the scalar damped wave equation

$$u_{tt} + \sigma u_t - \nabla \cdot (c^2 \nabla u) = f \qquad \text{in } \Omega \times (0,T)\,, \tag{3}$$

in a rectangular domain of size $[0,2] \times [0,1]$ with two rectangular barriers inside forming a narrow gap. Here $f(x,t)$ is a (known) source term, whereas the damping coefficient $\sigma(x) \geq 0$ and the speed of propagation $c(x) > 0$ are piecewise smooth. We use continuous P^2 elements on a triangular mesh, which is highly refined in the vicinity of the gap, as shown in Fig. 1. For the time discretization, we choose an LTS

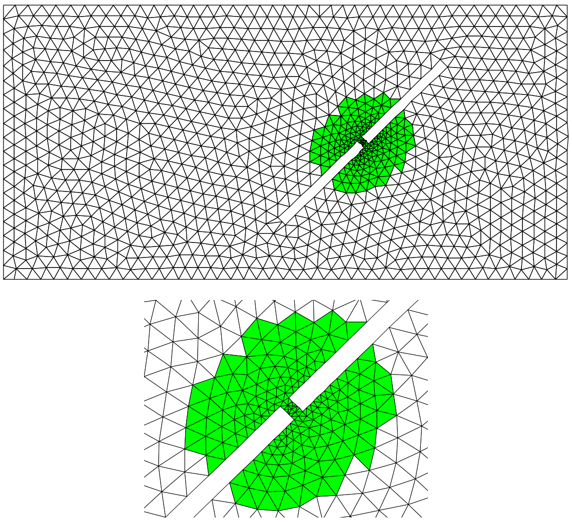

Fig. 1. The initial triangular mesh (left); zoom on the “fine” mesh indicated by the darker (green) triangles (right).

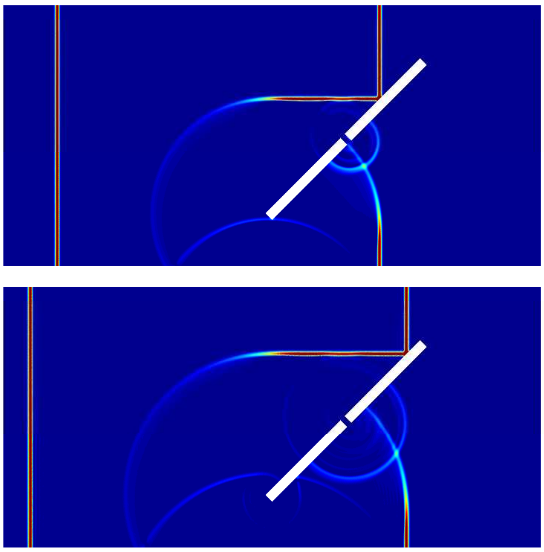

Fig. 2. The solution at times t =0.6 and 0.7.

method based on an explicit third-order low-storage Runge-Kutta scheme. Since the typical mesh size inside the refined region is about $p = 7$ times smaller than that in the surrounding coarser region, we take p local time steps of size $\Delta\tau = \Delta t/p$ for every time step Δt. Thus, the numerical method is third-order accurate both in space and time with respect to the L^2-norm. In Fig. 2, a Gaussian pulse initiates two plane waves, which propagate horizontally in opposite directions. As the right-moving wave impinges upon the obstacle, a small fraction of the wave penetrates the gap and generates multiple circular waves on both sides of the obstacle, which further interact with the wave field.

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