Field of values analysis of Laplace preconditioners for the Helmholtz equation.

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Summary. In this talk, we analyze the convergence of the preconditioned GMRES method for the first order finite element discretizations of the Helmholtz equation in media with losses. We consider a Laplace preconditioner and an inexact Laplace preconditioner. Our analysis is based on bounding the field of values of the preconditioned matrix in the complex plane. The obtained results are illustrated by numerical examples.

1 Introduction

Finite element discretizations of wave propagation problems lead to very large, indefinite, non-hermitian, and complex valued linear systems. One strategy to solve these systems is to use a suitable Krylov subspace solver such as GMRES, CGN, BiCGStab (see [6]) together with a preconditioner.

Finding good preconditioners for wave propagation problems has proven to be very difficult. The number of iterations required to solve the linear system depends strongly on the mesh density h and on the wave-number κ. For Helmholtz equation, the dependency between mesh density and the required number of iterations is due to the Laplace-operator part. Several preconditioners are capable of eliminating this dependency, see e.g. [3, 4]. The κ-dependency is related to the indefiniteness of the problem. Eliminating it has proven to be considerably more difficult.

Preconditioners for the Helmholtz equation can be divided roughly into shifted-Laplace (see e.g [3, 4]) and two-level methods (see e.g. [1, 5]). The shifted-Laplace preconditioners are successful in cutting the growth in the condition number due to the Laplace operator part. However, a κ-dependency in the required number of iterations still remains in the preconditioned system. The two-level preconditioners can eliminate this dependency, but are very expensive to evaluate.

In this talk, we consider the problem

$$\begin{aligned} -\Delta u - (\kappa^2 - \mathrm{i}\sigma)u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where $\kappa, \sigma \in \mathbb{R},\ \kappa > 0, \sigma > 0$. The domain $\Omega \subset \mathbb{R}^d, d = 2,3$ and the load function $f \in L^2(\Omega)$. We assume that this problem is discretized using first order finite elements on triangular of tetrahedral quasi-regular mesh. The mesh density parameter is denoted as h.

We present a field of values (FOV) based analysis for the convergence of the preconditioned GMRES method. We consider the Laplace and the inexact Laplace preconditioner, in which the Laplace problem is solved approximately by using an suitable iterative method.

A FOV analysis has been given in [8] for Hermitian positive definite split preconditioners and for shifted-Laplace preconditioner in [7]. The main difference compared to this work is that we estimate the FOV by using methods similar to the ones applied in the analysis of additive Schwarz preconditioners for elliptic problems. The novelty of our approach is that it allows us to analyze the inexact Laplace preconditioners in detail and it can also be applied to analyze two-level preconditioners.

The presented approach also takes the non-normal nature of the linear system automatically into account. This is especially important for inexact Laplace preconditioners, as their non-normality is not solely related to the mass matrix. We can analyze these preconditioners via an perturbation argument.

2 Field of values

The convergence of the GMRES method (see [6]) for the linear system $A\mathbf{x} = \mathbf{b}$ is related to the minimization problem

$$|\mathbf{r}_i| = \min_{\substack{p \in P_i \\ p(0)=1}} |p(A)\mathbf{r}_0|, \tag{2}$$

in which $\mathbf{r}_i$ is the residual on step i and P_i the space of polynomials of order i. Based on this minimization problem, different convergence estimates can be derived, see e.g. [2]. When the matrix A is no-normal, the convergence can be related to the properties of the pseudospectrum or the field of values (FOV).

The FOV is defined as the set

$$\mathscr{F}(A) = \left\{ \frac{\mathbf{x}^* A \mathbf{x}}{\mathbf{x}^* \mathbf{x}} \;\middle|\; \mathbf{x} \in \mathbb{C}^n,\ \mathbf{x} \neq 0 \right\}. \tag{3}$$

The convergence of GMRES is related to the dimensions and the location of the FOV in the complex plane. A simple estimate is given as

$$|\mathbf{r}_i| \le \left(\frac{s}{|c|}\right)^i |\mathbf{r}_0|. \tag{4}$$

In which s and c are related to the disk

$$D = \{ z \in \mathbb{C} \mid |z-c| \le s \}$$

containing the FOV, but not the origin.

3 Laplace preconditioner

The Laplace preconditioner $P : V_h \to V_h$ is defined as: For each $u \in V_h$ find $Pu \in V_h$ such that

$$(\nabla Pu, \nabla v) = (u, v) \quad \forall v \in V_h. \tag{5}$$

The matrix form of the operator P is $K^{-1}M$, where K is the stiffness matrix and M the mass matrix. The right preconditioned linear system is

$$AK^{-1}M\tilde{\mathbf{x}} = \mathbf{b}. \tag{6}$$

The FOV for this system is characterized by two following Theorems.

Theorem 1. *There exists a constant $C > 0$, independent of h, σ, and κ, such that*

$$\mathscr{F}(AK^{-1}M) \subset [C(1-\kappa^2)h^d, Ch^d] \times [0, C\sigma h^d],$$

in which d is the spatial dimension.

Theorem 2. *There exists a constant $C > 0$, independent of h, σ and κ, such that $\mathscr{F}(AK^{-1}M) \subset S$,*

$$S = \left\{ z \in \mathbb{C} \;\middle|\; ch^d - \frac{\kappa^2}{\sigma}\Im z \le \Re z \le Ch^d - \frac{\kappa^2}{\sigma}\Im z \right\},$$

in which d is the spatial dimension.

4 Inexact Laplace preconditioned

In practical computations, the solution to the Laplace problem $K\mathbf{x} = \mathbf{b}$ would be replaced with some approximation $\mathbf{x} \approx K_N^{-1}\mathbf{b}$.

We assume that such an approximation is obtained with a symmetric iterative method convergent in the $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$ norms. This is, there exists constants $\gamma_i, 0 < \gamma_i < 1, i = 1,2$ and $C > 0$, independent on γ_0 and γ_1, such that for any $u \in V_h$ there holds

$$\|E_N u\|_{H^1(\Omega)} \le C\gamma_1^N \|u\|_{H^1(\Omega)}$$

and

$$\|E_N u\|_{L^2(\Omega)} \le C\gamma_0^N \|u\|_{L^2(\Omega)}$$

In which, $E_N$is the error propagation operator relating e_0 to e_N, i.e., error on step 0 to error on step N. A suitable approximation can be obtained for example with the multigrid method.

The FOV for the preconditioned system satisfies

$$\mathscr{F}(A\tilde{K}_N^{-1}M) \subseteq \mathscr{F}(AK^{-1}M) \oplus \mathscr{F}(A(K_N^{-1} - K^{-1})M).$$

Bound for the FOV is obtained by combining an estimate for the size of the perturbation set

$$\mathscr{F}(A(K_N^{-1} - K^{-1})M). \tag{7}$$

with an estimate for the FOV for the Laplace preconditioned system.

Theorem 1. *There exists a constant $C > 0$, independent of γ_0, γ_1, κ, h, and σ, such that*

$$\Re\mathscr{F}(A(K_N^{-1} - K^{-1})M) \subset U_R$$

and

$$\Im\mathscr{F}(A(K_N^{-1} - K^{-1})M) \subset U_I$$

in which

$$U_R = \left[-Ch^d(\gamma_0^N + \kappa^2\gamma_1^N), Ch^d(\gamma_0^N + \kappa^2\gamma_1^N)\right]$$

and

$$U_I = \left[-Ch^d(\gamma_0^N + \sigma\gamma_1^N), Ch^d(\gamma_0^N + \sigma\gamma_1^N)\right].$$

where d is the spatial dimension and N the number of iterations used to compute the preconditioned.

From theoretical point of view, the implication of this theorem is that the number of iterations should be increased when the parameter κ grows to keep the size of the perturbation set small and the origin outside the FOV.

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